# A Modified Version of Free Orbit-Dimension of von Neumann Algebras

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**Abstract** Based on the notion of free orbit-dimension introduced by D. Hadwin and J. Shen [4], we introduce a new invariant on finite von Neumann algebras that do not necessarily act on separable Hilbert space. We show that this invariant is independent on the generating set, and we extend some results in [4] to von Neumann algebras that are not finitely generated.

## 1 Introduction

The theory of free entropy and free entropy dimension was developed by D. Voiculescu in the 1990's and it is one of the most powerful and exciting new tools in the theory of von Neumann algebras. D. Voiculescu [15] [16] introduced the concept of free entropy in relation to his free probability theory and the concept of free entropy dimension, and he used them to prove that the free group factors do not contain Cartan subalgebras, which answered a long-standing open problem. Later this was generalized by L. Ge [6], who showed that the free group factors do not contain a simple masa. L. Ge [7] used free entropy to give the first example of a separable prime $\mathrm{II}_1$ factor. Later, L. Ge and J. Shen [8] computed the free entropy dimension of some $\mathrm{II}_1$ factors with property T, including $\mathcal{L}(SL(\mathbb{Z}, 2m+1))$ $(m \leq 1)$. Recently, D. Hadwin and J. Shen [4] introduced a new invariant, the upper free orbit-dimension of a finite von Neumann algebra, which is closely related to Voiculescu's free entropy dimension. Using their new invariant, they generalized and simplified the proofs of most of the applications of free entropy dimension to finite von Neumann algebras.

Here we introduce a new invariant, $\mathfrak{K}_3$, which is a modification of the upper free orbit-dimension, $\mathfrak{K}_2$; when $\mathfrak{K}_2$ is defined, $\mathfrak{K}_3 = \infty \cdot \mathfrak{K}_2$. We then extend the domain of $\mathfrak{K}_3$ to all finite von Neumann algebras that can be embedded into some ultrapower of the hyperfinite $\mathrm{II}_1$ factors. This includes algebras acting on nonseparable Hilbert spaces.

The organization of the paper is as follows. In section 2, we recall the definition of free orbit-dimension, and introduce a new invariant $\mathfrak{K}_3$ on von Neumann algebras. In section 3, we prove:

(1)$\mathfrak{K}_3(\mathcal{S}) = \mathfrak{K}_3(\mathcal{G})$ when $W^*(\mathcal{S}) = W^*(\mathcal{G})$ (independence of the generators),

(2) if $\mathcal{N}_1 \cap \mathcal{N}_2$ is diffuse, then $\mathfrak{K}_3\left((\mathcal{N}_1 \cup \mathcal{N}_2)''\right) \leq \mathfrak{K}_3(\mathcal{N}_1) + \mathfrak{K}_3(\mathcal{N}_2)$,

(3) $\mathfrak{K}_3\left(W^*(\mathcal{N} \cup \{y\})\right) \leq \mathfrak{K}_3(\mathcal{N})$ whenever there exist normal operators $a$ and $b$ in $\mathcal{N}$ without common eigenvalues such that $ay = yb \neq 0$.

In section 4 we prove:

(1) if $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ is a family of von Neumann algebras such that each $\mathcal{M}_\lambda$ has a central net of Haar untiaries, and $\alpha$ is a nontrivial ultrafilter on $\Lambda$, then $\mathfrak{K}_3(\prod^\alpha \mathcal{M}_\lambda) = 0$,

(2) if $\mathcal{M}$ is a von Neumann algebra with a central net of Haar unitaries, then $\mathfrak{K}_3(\mathcal{M}) = 0$,

(3) if $\mathbb{F}$ is a free group with the standard generating set $G$ satisfying $|G| \geq 2$, then $\mathfrak{K}_3(\mathcal{L}_\mathbb{F}) = \infty$,

(4) if $\mathcal{N}_1$ and $\mathcal{N}_2$ are mutually commuting diffuse subalgebras of $\mathcal{M}$, then $\mathfrak{K}_3(W^*(\mathcal{N}_1 \cup \mathcal{N}_2)) = 0$,

(5) if $\mathcal{M}$ is a $\mathrm{II}_1$ factor and $\mathfrak{K}_3(\mathcal{M}) = \infty$, then $\mathcal{M}$ is prime (i.e., cannot be written as a tensor product of two $\mathrm{II}_1$ factors).

In section 5, we show how our invariant leads naturally to a canonical decomposition of torsion-free groups into a union of certain self-normalizing subgroups so that the intersection of any two of them is $\{e\}$. We completely describe this decomposition for free groups, and we present a related question for the free group factor $\mathcal{L}_{\mathbb{F}_2}$.

All of the free entropy concepts require the von Neumann algebra $\mathcal{M}$ under consideration can be tracially embedded into an ultrapower of the hyperfinite $\mathrm{II}_1$ factor. Throughout this paper, we assume that all the von Neumann algebras we consider can be embedded.

# 2 Preliminaries

First we recall the definition of $\mathfrak{K}_2$ introduced by D. Hadwin and J. Shen [4], then we introduce our new invariant $\mathfrak{K}_3$.

Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in $\mathbb{C}$, and $\tau_k$ be the normalized trace on $\mathcal{M}_k(\mathbb{C})$, i.e., $\tau_k = \frac{1}{k} Tr$, where $Tr$ is the usual trace on $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{U}_k$ be the group of all unitary matrices in $\mathcal{M}_k(\mathbb{C})$ and $\mathcal{M}_k(\mathbb{C})^n$ denote the direct sum of $n$ copies of $\mathcal{M}_k(\mathbb{C})$. Define $\|\cdot\|_2$ on $\mathcal{M}_k(\mathbb{C})^n$ by

$$\|(A_1, \ldots, A_n)\|_2^2 = \tau_k(A_1^* A_1) + \ldots + \tau_k(A_n^* A_n)$$

for all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$.

For every $\omega > 0$, define the *$\omega$-ball* $Ball(B_1, \ldots, B_n; \omega)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that

$$\|(A_1, \ldots, A_n) - (B_1, \ldots, B_n)\|_2 < \omega.$$

Define the *$\omega$-orbit-ball* $\mathcal{U}(B_1, \ldots, B_n; \omega)$ in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that there exists some unitary matrix $W$ in $\mathcal{U}_k$ satisfying

$$\|(A_1, \ldots, A_n) - (W B_1 W^*, \ldots, W B_n W^*)\|_2 < \omega.$$

Suppose $E \subseteq \mathcal{M}_k(\mathbb{C})^n$, $\omega > 0$. Define the *covering number* $\nu_2(E, \omega)$ to be the minimal number of $\omega$-balls that cover $E$ with the centers of these $\omega$-balls in $E$; define the *$\omega$-orbit covering number* $\nu(E, \omega)$ to be the minimal number of $\omega$-orbit-balls that cover $E$ with the centers of these $\omega$-orbit-balls in $E$.

Let $\mathcal{M}$ be a von Neumann algebra with a tracial state $\tau$ and $x_1, x_2, \ldots, x_n$ be elements in $\mathcal{M}$. For any $R, \varepsilon > 0$, and positive integers $m$ and $k$, define $\Gamma_R(x_1, \ldots, x_n; m, k, \varepsilon)$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all $(A_1, \ldots, A_n)$ in $\mathcal{M}_k(\mathbb{C})^n$ such that $\|A_j\| \le R$ for $1 \le j \le n$, and

$$|\tau_k(A_{i_1}^{\eta_1} \cdots A_{i_q}^{\eta_q}) - \tau(x_{i_1}^{\eta_1} \cdots x_{i_q}^{\eta_q})| < \varepsilon,$$

for all $1 \le i_1, \ldots, i_q \le n$, all $\eta_1, \ldots, \eta_q \in \{1, *\}$ and all $q$ with $1 \le q \le m$.

Define

$$\mathfrak{K}(x_1,\ldots,x_n;m,\varepsilon,\omega,R)=\limsup_{k\to\infty}\frac{\log(\nu(\Gamma_R(x_1,\ldots,x_n;m,k,\varepsilon),\omega))}{-k^2\log\omega}$$

$$\mathfrak{K}(x_1,\ldots,x_n;\omega,R)=\inf_{m\in\mathbb{N},\varepsilon>0}\mathfrak{K}(x_1,\ldots,x_n;m,\varepsilon,\omega,R)$$

$$\mathfrak{K}(x_1,\ldots,x_n;\omega)=\sup_{R>0}\mathfrak{K}(x_1,\ldots,x_n;\omega,R)$$

$$\mathfrak{K}_2(x_1,\ldots,x_n)=\sup_{0<\omega<1}\mathfrak{K}(x_1,\ldots,x_n;\omega)$$

D. Hadwin and J. Shen [4] also defined $\mathfrak{K}_2(x_1,\ldots,x_n:y_1,\ldots,y_p)$ for all $x_1,\ldots,x_n$, $y_1,\ldots,y_p$ in the von Neumann algebra $\mathcal{M}$ as follows. Let

$$\begin{aligned}&\Gamma_R(x_1,\ldots,x_n:y_1,\ldots,y_p;m,k,\varepsilon)\\ =\ &\{(A_1,\ldots,A_n)\in\mathcal{M}_k(\mathbb{C})^n:\ \text{there exist } B_1,\ldots,B_p \text{ in } \mathcal{M}_k(\mathbb{C})\\ &\text{such that } (A_1,\ldots,A_n,B_1,\ldots,B_p)\in\Gamma_R(x_1,\ldots,x_n,y_1,\ldots,y_p;m,k,\varepsilon)\}\,,\end{aligned}$$

$$\mathfrak{K}(x_1,\ldots,x_n:y_1,\ldots,y_p;m,\varepsilon,\omega,R)=\limsup_{k\to\infty}\frac{\log(\nu(\Gamma_R(x_1,\ldots,x_n;m,k,\varepsilon),\omega))}{-k^2\log\omega}$$

$$\mathfrak{K}(x_1,\ldots,x_n:y_1,\ldots,y_p;\omega,R)=\inf_{m\in\mathbb{N},\varepsilon>0}\mathfrak{K}(x_1,\ldots,x_n:y_1,\ldots,y_p;m,\varepsilon,\omega,R)$$

$$\mathfrak{K}(x_1,\ldots,x_n:y_1,\ldots,y_p;\omega)=\sup_{R>0}\mathfrak{K}(x_1,\ldots,x_n;\omega,R)$$

$$\mathfrak{K}_2(x_1,\ldots,x_n:y_1,\ldots,y_p)=\sup_{0<\omega<1}\mathfrak{K}(x_1,\ldots,x_n;\omega)$$

**Remark 2.1** *From the definition, it is clear that*
*(1)* $\mathfrak{K}_2(x_1,\ldots,x_n:y_1,\ldots,y_p)\geq\mathfrak{K}_2(x_1,\ldots,x_n:y_1,\ldots,y_p,y_{p+1})$,
*(2) if* $\mathfrak{K}_2(x_1,\ldots,x_n:x_1,\ldots,x_{n+j})=0$ $(j\geq 0)$*, then*

$$\mathfrak{K}_2(x_1,\ldots,x_{n-1}:x_1,\ldots,x_{n+j})=0.$$

Let $\infty\cdot 0=0$. For any subset $\mathcal{G}$ of $\mathcal{M}$, define

$$\mathfrak{K}_3(x_1,\ldots,x_n:\mathcal{G})=\inf\{\infty\cdot\mathfrak{K}_2(x_1,\ldots,x_n:A):\ A \text{ is a finite subset of } \mathcal{G}\}\,,$$

$$\mathfrak{K}_3(\mathcal{G})=\sup_{\substack{E\subseteq\mathcal{G}\\ E\text{ is finte}}}\ \inf_{\substack{F\subseteq\mathcal{G}\\ F\text{ is finte}}}\ \infty\cdot\mathfrak{K}_2(E:F).$$

When $\mathcal{G}$ is finite, it is not difficult to see that

$$\mathfrak{K}_3(x_1,\ldots,x_n:\mathcal{G})=\infty\cdot\mathfrak{K}_2(x_1,\ldots,x_n:\mathcal{G})$$

and

$$\mathfrak{K}_3(\mathcal{G})=\infty\cdot\mathfrak{K}_2(\mathcal{G}).$$

Note that the value of $\mathfrak{K}_3(x_1,\ldots,x_n:y_1,\ldots,y_p)$ or $\mathfrak{K}_3(x_1,\ldots,x_n)$ is always 0 or $\infty$.

## 3 Key properties of $\mathfrak{K}_3$

**Theorem 3.1** *If $\mathcal{M}$ is a von Neumann algebra with a tracial state $\tau$, then the following are equivalent:*

*(1) $\mathfrak{K}_3(\mathcal{M}) = 0$;*

*(2) if $x_1, \ldots, x_n \in \mathcal{M}$, then there exist $y_1, \ldots, y_t \in \mathcal{M}$ such that $\mathfrak{K}_2(x_1, \ldots, x_n : y_1, \ldots, y_t) = 0$;*

*(3) for any generating set $\mathcal{G}$ of $\mathcal{M}$, $\mathfrak{K}_3(\mathcal{G}) = 0$;*

*(4) there exists a generating set $\mathcal{G}$ of $\mathcal{M}$ such that $\mathfrak{K}_3(\mathcal{G}) = 0$;*

*(5) if $\mathcal{G}$ is a generating set of $\mathcal{M}$, and $A_0$ is a finite subset of $\mathcal{G}$, then, for any finite subset $A$ with $A_0 \subseteq A \subseteq \mathcal{G}$, there exists a finite subset $B$ of $\mathcal{G}$ so that $\mathfrak{K}_2(A : B) = 0$;*

*(6) there is an increasing directed family $\{\mathcal{M}_\iota : \iota \in \Lambda\}$ of von Neumann subalgebras of $\mathcal{M}$ such that*

*(a) each $\mathcal{M}_\iota$ is countably generated,*

*(b) $\mathfrak{K}_3(\mathcal{M}_\iota) = 0$,*

*(c) $\mathcal{M} = \cup_{\iota \in \Lambda} \mathcal{M}_\iota$.*

Proof. It is clear that (1)$\Leftrightarrow$(2), (3)$\Rightarrow$(4) and (3)$\Rightarrow$(5).

(4)$\Rightarrow$(2) Suppose $\mathcal{G}$ is a generating set of $\mathcal{M}$ and $\mathfrak{K}_3(\mathcal{G}) = 0$. Let $\omega > 0$ and $x_1, \ldots, x_n$ be any elements in $\mathcal{M}$. Then there exist polynomials $p_1, \ldots, p_n$ and elements $y_1, \ldots, y_s$ in $\mathcal{G}$ such that

$$\|(x_1, \ldots, x_n) - (p_1(y_1, \ldots, y_s), \ldots, p_n(y_1, \ldots, y_s))\| \leq \frac{\omega}{4}.$$

For any given $R > 0$, if $(A_1, \ldots, A_s), (B_1, \ldots, B_s)$ in $\mathcal{M}_k(\mathbb{C})^s$ and $\|A_j\|, \|B_j\| \leq R$ for all $1 \leq j \leq s$, then there exists a positive integer $N$ such that

$$\begin{aligned} & \|(p_1(A_1, \ldots, A_s), \ldots, p_n(A_1, \ldots, A_s)) - (p_1(B_1, \ldots, B_s), \ldots, p_n(B_1, \ldots, B_s))\|_2 \\ \leq \ & N\|(A_1, \ldots, A_s) - (B_1, \ldots, B_s)\|_2. \end{aligned}$$

Since $\mathfrak{K}_3(\mathcal{G}) = 0$ and $y_1, \ldots, y_m$ are in $\mathcal{G}$, there exist $y_1, \ldots, y_t$ $(t \geq s)$ in $\mathcal{G}$ such that

$$\mathfrak{K}_2(y_1, \ldots, y_s : y_1, \ldots, y_t) = 0.$$

Let $R > \max\{\|y_i\|, \|x_j\| : 1 \leq i \leq n, 1 \leq j \leq t\}$. Note that if $(A_1, \ldots, A_n, B_1, \ldots, B_t) \in \Gamma_R(x_1, \ldots, x_n, y_1, \ldots, y_t; m, k, \varepsilon)$, then, for sufficiently small $\varepsilon$ and sufficiently large $m$, we have

$$\|(A_1, \ldots, A_n) - (p_1(B_1, \ldots, B_s), \ldots, p_n(B_1, \ldots, B_s))\|_2 < \frac{\omega}{4}$$

and

$$(B_1, \ldots, B_s) \in \Gamma(y_1, \ldots, y_s : y_1, \ldots, y_t; m, k, \varepsilon).$$

It follows that there is a set $\Lambda$ and a subset $\{(B_1^\lambda, \ldots, B_s^\lambda) : \lambda \in \Lambda\}$ of $\Gamma_R(y_1, \ldots, y_s : y_1, \ldots, y_t; m, k, \varepsilon)$ with

$$\mathrm{card}(\Lambda) \leq \nu(\Gamma(y_1, \ldots, y_s : y_1, \ldots, y_t; m, k, \varepsilon), \frac{\omega}{4N}).$$

That means, for every $(A_1, \ldots, A_n, B_1, \ldots, B_t) \in \Gamma_R(x_1, \ldots, x_n, y_1, \ldots, y_t; m, k, \varepsilon)$, there is a $\lambda \in \Lambda$ and a unitary $k \times k$ matrix $U$ such that

$$\|(B_1, \ldots, B_s) - U^*(B_1^\lambda, \ldots, B_s^\lambda)U\|_2 \leq \frac{\omega}{4N}.$$

That gives

$$\|(A_1,\ldots A_n)-U^*(p_1(B_1^\lambda,\ldots,B_s^\lambda),\ldots,p_n(B_1^\lambda,\ldots,B_s^\lambda))U\|_2<\frac{\omega}{2}.$$

It follows that, if $\varepsilon$ is sufficient small and $m$ is sufficient large, then for any $k\in\mathbb{N}$,

$$\begin{aligned}&\frac{\nu(\Gamma_R(x_1,\ldots,x_n:y_1,\ldots,y_t;m,k,\varepsilon),\omega)}{-k^2\log\omega}\\ \leq\ &\left(\frac{\log\omega}{\log(\omega/4N)}\right)\frac{\nu(\Gamma(y_1,\ldots,y_s:y_1,\ldots,y_t;m,k,\varepsilon),\omega/4N)}{-k^2\log(\omega/4N)}.\end{aligned}$$

If we take $k\to\infty$ and take the infimum over $m$ and $\varepsilon$, we get

$$\begin{aligned}&\mathfrak{K}_2(x_1,\ldots,x_n:y_1,\ldots,y_t;\omega)\\ \leq\ &\left(\frac{\log\omega}{\log(\omega/4N)}\right)\mathfrak{K}_2(y_1,\ldots,y_s:y_1,\ldots,y_t)\\ =\ &0.\end{aligned}$$

Hence, we have $\mathfrak{K}_2(x_1,\ldots,x_n:y_1,\ldots,y_t)=0$.

(2)$\Rightarrow$(3) Suppose (2) is true, and suppose $x_1,\ldots,x_n$ are elements of some generating set $\mathcal{G}$ of $\mathcal{M}$. Then there exist $y_1,\ldots,y_t$ in $\mathcal{M}$ such that $\mathfrak{K}_2(x_1,\ldots,x_n:y_1,\ldots,y_t)=0$.

Suppose $\varepsilon_0>0$, $m_0\in\mathbb{N}$, and $0<\omega<1$. We can choose $w_1,\ldots,w_s$ in $\mathcal{G}$ and polynomials $p_1,\ldots,p_t$ so that each $\|y_j-p_j(w_1,\ldots,w_s)\|_2$ $(1\leq j\leq t)$ is small enough to make

$$|\tau(q(x_1,\ldots,x_n,y_1,\ldots,y_t))-\tau(q(x_1,\ldots,x_n,p_1(w_1,\ldots,w_s),\ldots,p_t(w_1,\ldots,w_s)))|<\frac{\varepsilon_0}{4},$$

for every monomial $q$ with length at most $m_0$.

When $m$ is sufficient large, $\varepsilon$ is sufficient small, if

$$(A_1,\ldots,A_n,B_1,\ldots,B_s)\in\Gamma_R(x_1,\ldots,x_n,w_1,\ldots,w_s;m,k,\varepsilon),$$

then

$$(A_1,\ldots,A_n,p_1(B_1,\ldots,B_s),\ldots,p_t(B_1,\ldots,B_s))\in\Gamma_R(x_1,\ldots,x_n,y_1,\ldots,y_t;m_0,k,\varepsilon_0).$$

Hence

$$\Gamma_R(x_1,\ldots,x_n:w_1,\ldots,w_s;m,k,\varepsilon)\subseteq\Gamma_R(x_1,\ldots,x_n:y_1,\ldots,y_t;m_0,k,\varepsilon_0).$$

Since

$$\nu(\Gamma_R(x_1,\ldots,x_n:w_1,\ldots,w_s;m,k,\varepsilon),\omega)\leq 2\nu(\Gamma_R(x_1,\ldots,x_n:y_1,\ldots,y_t;m_0,k,\varepsilon_0),\omega),$$

we have,

$$\mathfrak{K}(x_1,\ldots,x_n:w_1,\ldots,w_s;\omega)\leq 2\mathfrak{K}(x_1,\ldots,x_n:y_1,\ldots,y_t;m_0,\varepsilon_0,\omega).$$

Then we get

$$\mathfrak{K}(x_1,\ldots,x_n:w_1,\ldots,w_s;\omega)\leq 2\mathfrak{K}(x_1,\ldots,x_n:y_1,\ldots,y_t;\omega).$$

Therefore

$$\mathfrak{K}_2(x_1,\ldots,x_n:w_1,\ldots,w_s)\leq 2\mathfrak{K}_2(x_1,\ldots,x_n:y_1,\ldots,y_t)=0.$$

Thus $\mathfrak{K}_2(x_1,\ldots,x_n : w_1,\ldots,w_s) = 0$. From the definition, $\mathfrak{K}_3(\mathcal{G}) = 0$.

(5)⇒(6) Suppose $A_0 \subseteq \{x_1,\ldots,x_n\} \subseteq \mathcal{G}$. From (4), there exists a family $\{B_0, B_1,\ldots\}$ of finite subsets of $\mathcal{G}$ such that

$$\mathfrak{K}_2(x_1,\ldots,x_n : B_0) = 0,$$

and for any positive integer $n$,

$$\mathfrak{K}_2(x_1,\ldots,x_n, B_0,\ldots,B_{n-1} : B_n) = 0.$$

Let $\mathcal{G}$ be the set $\{x_1,\ldots,x_n\} \cup \cup_{n=0}^{\infty} B_n$ and $\mathcal{N}$ be the von Neumann subalgebra generated by $\mathcal{G}$. Then $\mathcal{N}$ is countably generated.

Let $A \subseteq \mathcal{G}$ be a finite subset. Then there exists a positive integer $m$, so that $A \subseteq \{x_1,\ldots,x_n\} \cup B_0 \cup \cdots \cup B_m$. Since

$$\mathfrak{K}_2(x_1,\ldots,x_n, B_0,\ldots,B_m : B_{m+1}) = 0,$$

by Remark 2.1, we have

$$\mathfrak{K}_2(A : B_n) = 0.$$

It follows that $\mathfrak{K}_3(\mathcal{G}) = 0$. Therefore $\mathfrak{K}_3(\mathcal{N}) = 0$ by the equivalence of (1) and (4).

It is not difficult to see that the union of all such $\mathcal{N}$'s is $\mathcal{M}$.

(6)⇒(2) Suppose $x_1,\ldots,x_n$ are elements of $\mathcal{M}$. From (5), there exists $\{\iota_1,\ldots,\iota_n\} \subseteq \Lambda$ such that $x_1 \in \mathcal{M}_{\iota_1},\ldots,x_n \in \mathcal{M}_{\iota_n}$. Since $\{\mathcal{M}_\iota : \iota \in \Lambda\}$ is an increasing directed family, there exists $\iota \in \Lambda$, such that $\mathcal{M}_{\iota_1},\ldots,\mathcal{M}_{\iota_n} \subseteq \mathcal{M}_\iota$ and $\mathfrak{K}_3(\mathcal{M}_\iota) = 0$. Therefore there exist $y_1,\ldots,y_m \in \mathcal{M}_\iota \subseteq \mathcal{M}$ such that $\mathfrak{K}_2(x_1,\ldots,x_n : y_1,\ldots,y_m) = 0$. □

**Remark 3.2** *If $\mathcal{M}$ is finite generated, then $\mathfrak{K}_3(\mathcal{M}) = 0$ is equivalent to $\mathfrak{K}_2(\mathcal{M}) = 0$.*

**Corollary 3.3** *Suppose $\mathcal{M}$ is a von Neumann algebra with a tracial state $\tau$, $\mathcal{G}$ is a generating set of $\mathcal{M}$. Then $\mathfrak{K}_3(\mathcal{M}) = \mathfrak{K}_3(\mathcal{G})$.*

**Corollary 3.4** *Suppose $\{\mathcal{M}_\iota\}_{\iota\in\Lambda}$ is an increasingly directed family of von Neumann algebras. Then $\mathfrak{K}_3(\cup_\iota \mathcal{M}_\iota) \le \liminf_\iota \mathfrak{K}_3(\mathcal{M}_\iota)$.*

**Remark 3.5** *To see that Corollary 3.4 gives the best estimate, note that $\mathcal{L}_{\mathbb{F}_2} \otimes \mathcal{R} = \cup_n (\mathcal{L}_{\mathbb{F}_2} \otimes \mathcal{M}_{2^n}(\mathbb{C}))$ and $\mathfrak{K}_3(\mathcal{L}_{\mathbb{F}_2} \otimes \mathcal{R}) = 0$, but $\mathfrak{K}_3(\mathcal{L}_{\mathbb{F}_2} \otimes \mathcal{M}_{2^n}(\mathbb{C})) = \infty$ for every $n$.*

To prove Theorem 3.12, we need the following lemmas.

**Lemma 3.6** *Let $\mathcal{M}$ be a von Neumann algebra with a tracial state $\tau$. Suppose $x_1$, …, $x_n$, $y_1$, …, $y_p$, $w_1,\ldots,w_t$ are elements of $\mathcal{M}$ and $x_1,\ldots,x_n \in W^*(y_1,\ldots,y_p)$. Then, for $\omega > 0$,*

$$\mathfrak{K}(y_1,\ldots,y_p : w_1,\ldots,w_t; \omega) = \mathfrak{K}(y_1,\ldots,y_p : x_1,\ldots,x_n, w_1,\ldots,w_t; \omega).$$

Proof. It is not hard to get the "≥" part.

Assume $\varepsilon_0 > 0$, $m_0 \in \mathbb{N}$, $R > 1$. Since $x_1,\ldots,x_n \in W^*(y_1,\ldots,y_p)$, there exist $m_1 \in \mathbb{N}$ and $\varepsilon_1 > 0$ and a family of noncommutative polynomials $q_1,\ldots,q_n$ such that

$\| (q_1(y_1,\ldots,y_p),\ldots,q_n(y_1,\ldots,y_p)) - (x_1,\ldots,x_n)\|_2$ is so small that for any $m \geq m_1$ and $0 < \varepsilon \leq \varepsilon_1$, we have, for any $k \in \mathbb{N}$,

$$\begin{aligned} & \{(A_1,\ldots,A_p,q_1(A_1,\ldots,A_p),\ldots,q_n(A_1,\ldots,A_p),C_1,\ldots,C_t) : \\ & (A_1,\ldots,A_p,C_1,\ldots,C_t) \in \Gamma_R(y_1,\ldots,y_p,w_1,\ldots,w_t;m,k,\varepsilon)\} \\ \subseteq\ & \Gamma_R(y_1,\ldots,y_p,x_1,\ldots,x_n,w_1,\ldots,w_t;m_0,k,\varepsilon_0), \end{aligned}$$

which implies

$$\Gamma_R(y_1,\ldots,y_p : w_1,\ldots,w_t;m,k,\varepsilon) \subseteq \Gamma_R(y_1,\ldots,y_p : x_1,\ldots,x_n,w_1,\ldots,w_t;m_0,k,\varepsilon_0).$$

Therefore $\mathfrak{K}(y_1,\ldots,y_p : w_1,\ldots,w_t;\omega) \leq \mathfrak{K}(y_1,\ldots,y_p : x_1,\ldots,x_n,w_1,\ldots,w_t;\omega)$. □

The following lemma is a slight extension of Theorem 1 in [4]; the proofs are similar.

**Lemma 3.7** *Let $x_1,\ldots,x_n,y_1,\ldots,y_p,w_1,\ldots,w_t$ be elements in a von Neumann algebra $\mathcal{M}$ with a tracial state $\tau$, and $W^*(x_1,\ldots,x_n) = W^*(y_1,\ldots,y_p)$. Then*

$$\mathfrak{K}_3(x_1,\ldots,x_n : w_1,\ldots,w_t) = \mathfrak{K}_3(y_1,\ldots,y_p : w_1,\ldots,w_t).$$

Proof. If $\mathfrak{K}_3(x_1,\ldots,x_n : w_1,\ldots,w_t)$ and $\mathfrak{K}_3(y_1,\ldots,y_p : w_1,\ldots,w_t)$ are both infinity, then they are equal. If one of them is zero, say it $\mathfrak{K}_3(x_1,\ldots,x_n : w_1,\ldots,w_t)$, then we need to prove that $\mathfrak{K}_3(y_1,\ldots,y_p : w_1,\ldots,w_t) = 0$.

For every $0 < \omega < 1$, there exists a family of noncommutative polynomials $q_1,\ldots,q_p$, such that

$$\|(y_1,\ldots,y_p) - (q_1(x_1,\ldots,x_n),\ldots,q_p(x_1,\ldots,x_n))\|_2 \leq \frac{\omega}{4}.$$

For such a family of polynomials $q_1,\ldots,q_p$ and every $R > 0$, there always exists a constant $D \geq 1$, depending only on $q_1,\ldots,q_p$ and $R$, such that

$$\begin{aligned} & \| (q_1(A_1,\ldots,A_n),\ldots,q_p(A_1,\ldots,A_n)) - (q_1(B_1,\ldots,B_n),\ldots,q_p(B_1,\ldots,B_n)) \|_2 \\ \leq\ & D\|(A_1,\ldots,A_n) - (B_1,\ldots,B_n)\|_2, \end{aligned}$$

for all $A_1,\ldots,A_n,B_1,\ldots,B_n$ in $\mathcal{M}_k(\mathbb{C})$ satisfying $\|A_j\|,\|B_j\| \leq R$, for $1 \leq j \leq n$.

For $R > 1$, $m$ and $k$ sufficiently large, $\varepsilon$ sufficiently small, if

$$(B_1,\ldots,B_p,A_1,\ldots,A_n) \in \Gamma_R(y_1,\ldots,y_p,x_1,\ldots,x_n : w_1,\ldots,w_t;m,k,\varepsilon),$$

then

$$\|(B_1,\ldots,B_p) - (q_1(A_1,\ldots,A_n),\ldots,q_p(A_1,\ldots,A_n))\|_2 \leq \frac{\omega}{4}.$$

It is clear that $(A_1,\ldots,A_n) \in \Gamma_R(x_1,\ldots,x_n : w_1,\ldots,w_t;m,k,\varepsilon)$.

There exists a set $\{\mathcal{U}(A_1^\lambda,\ldots,A_n^\lambda;\frac{\omega}{4D})\}_{\lambda\in\Lambda_k}$ of $\frac{\omega}{4D}$-orbit-balls that cover $\Gamma_R(x_1,\ldots,x_n :$
$w_1,\ldots,w_t;$
$m,k,\varepsilon)$ with the cardinality of $\Lambda_k$ satisfying $|\Lambda_k| = \nu(\Gamma_R(x_1,\ldots,x_n : w_1,\ldots,w_t;m,k,\varepsilon),\frac{\omega}{4D})$. Thus there exists some $\lambda \in \Lambda_k$ and $U \in \mathcal{U}_k$ such that

$$\|(A_1,\ldots,A_n) - U(A_1^\lambda,\ldots,A_n^\lambda)U^*\|_2 \leq \frac{\omega}{4D}.$$

It follows that

$$\begin{aligned}
& \|(B_1,\ldots,B_p) - U\left(q_1(A_1^\lambda,\ldots,A_n^\lambda),\ldots,q_p(A_1^\lambda,\ldots,A_n^\lambda)\right)U^*\|_2 \\
= & \|(B_1,\ldots,B_p) - \left(q_1(U(A_1^\lambda,\ldots,A_n^\lambda)U^*),\ldots,q_p(U(A_1^\lambda,\ldots,A_n^\lambda)U^*)\right)\|_2 \\
\leq & \frac{\omega}{2}.
\end{aligned}$$

That is,

$$(B_1,\ldots,B_p) \in \mathcal{U}\left(q_1(A_1^\lambda,\ldots,A_n^\lambda),\ldots,q_p(A_1^\lambda,\ldots,A_n^\lambda);\omega\right).$$

Hence, we get

$$\begin{aligned}
0 & \leq \mathfrak{K}(y_1,\ldots,y_p : x_1,\ldots,x_n,w_1,\ldots,w_t;\omega,R) \\
& \leq \inf_{m\in\mathbb{N},\varepsilon>0} \limsup_{k\to\infty} \frac{\log(|\Lambda_k|)}{-k^2\log\omega} \\
& = \inf_{m\in\mathbb{N},\varepsilon>0} \limsup_{k\to\infty} \frac{\log(\nu(\Gamma_R(x_1,\ldots,x_n : w_1,\ldots,w_t;m,k,\varepsilon),\frac{\omega}{4D}))}{-k^2\log\omega} \\
& = 0.
\end{aligned}$$

Therefore $\mathfrak{K}_2(y_1,\ldots,y_p : x_1,\ldots,x_n,w_1,\ldots,w_t) = 0$. From Lemma 3.6, we get $\mathfrak{K}_2(y_1,\ldots,y_p : w_1,\ldots,w_t) = 0$. So $\mathfrak{K}_3(y_1,\ldots,y_p : w_1,\ldots,w_t) = 0$. □

**Definition 3.8** *A unitary matrix $A$ in $\mathcal{M}_k(\mathbb{C})$ is called a Haar unitary matrix if the eigenvalues of $A$ are the $k$-th roots of unity; equivalently, if $\tau_k(A^i) = 0$ for $1 \leq i < k$ and $\tau_k(A^k) = 1$.*

**Lemma 3.9 ([16])** *Let $V_1, V_2$ be two Haar unitary matrices in $\mathcal{M}_k(\mathbb{C})$. For every $\delta > 0$, let*

$$\Omega(V_1,V_2;\delta) = \{U \in \mathcal{U}_k : \|UV_1 - V_2U\|_2 \leq \delta\}.$$

*Then, for every $0 < \delta < r$, $\nu_2(\Omega(V_1,V_2;\delta),\frac{4\delta}{r}) \leq (\frac{3r}{2\delta})^{4rk^2}$.*

**Definition 3.10** *Suppose $\mathcal{M}$ is a von Neumann algebra with a tracial state $\tau$. Then a unitary $u$ in $\mathcal{M}$ is called a Haar unitary if $\tau(u^m) = 0$ when $m \neq 0$. In addition, $\mathcal{M}$ is called diffuse if $\mathcal{M}$ contains a Haar unitary.*

The following lemma is a slight extension of Theorem 6 in [4], and the proofs are similar.

**Lemma 3.11** *Let $x_1,\ldots,x_n,y_1,\ldots,y_p,v_1,\ldots,v_s,w_1,\ldots,w_t$ be elements in a von Neumann algebra $\mathcal{M}$ with a tracial state $\tau$. If $W^*(x_1,\ldots,x_n) \cap W^*(y_1,\ldots,y_p)$ is diffuse, then*

$$\begin{aligned}
& \mathfrak{K}_3(x_1,\ldots,x_n,y_1,\ldots,y_p : v_1,\ldots,v_s,w_1,\ldots,w_t) \\
\leq & \mathfrak{K}_3(x_1,\ldots,x_n : v_1,\ldots,v_s) + \mathfrak{K}_3(y_1,\ldots,y_p : w_1,\ldots,w_t).
\end{aligned}$$

Proof. If one of $\mathfrak{K}_3(x_1,\ldots,x_n : v_1,\ldots,v_s)$ and $\mathfrak{K}_3(y_1,\ldots,y_p : w_1,\ldots,w_t)$ is infinity, then we are done.

Now suppose $\mathfrak{K}_3(x_1,\ldots,x_n : v_1,\ldots,v_s) = \mathfrak{K}_3(y_1,\ldots,y_p : w_1,\ldots,w_t) = 0$. Since $W^*(x_1,\ldots,x_n) \cap W^*(y_1,\ldots,y_p)$ is diffuse, we can find a Haar unitary $u$ in $W^*(x_1,\ldots,x_n) \cap W^*(y_1,\ldots,y_p)$.

For $R > 1 + \max_{1\le i\le n, 1\le j\le p}\{\|x_i\|, \|y_j\|\}$, $0 < \omega < \frac{1}{2n}$, $0 < r < 1$ and $\varepsilon > 0$, $m, k \in \mathbb{N}$. Suppose

$$(A_1, \ldots, A_n, B_1, \ldots, B_p, U) \in \Gamma_R(x_1, \ldots, x_n, y_1, \ldots, y_p, u : v_1, \ldots, v_s, w_1, \ldots, w_t; m, k, \varepsilon).$$

Then

$$(A_1, \ldots, A_n, U) \in \Gamma_R(x_1, \ldots, x_n, u : v_1, \ldots, v_s, w_1, \ldots, w_t; m, k, \varepsilon)$$

and

$$(B_1, \ldots, B_p, U) \in \Gamma_R(y_1, \ldots, y_p, u : v_1, \ldots, v_s, w_1, \ldots, w_t; m, k, \varepsilon).$$

Let $\{\mathcal{U}(A_1^\lambda, \ldots, A_n^\lambda, U^\lambda); \frac{r\omega}{24R}\}_{\lambda\in\Lambda_k}$ be a set of $\frac{r\omega}{24R}$-orbit-balls that cover $\Gamma_R(x_1, \ldots, x_n, u :$ $v_1, \ldots, v_s,$
$w_1, \ldots, w_t; m, k, \varepsilon)$ with the cardinality of $\Lambda_k$ satisfying

$$|\Lambda_k| = \nu(\Gamma_R(x_1, \ldots, x_n, u : v_1, \ldots, v_s, w_1, \ldots, w_t; m, k, \varepsilon); \frac{r\omega}{24R}).$$

Also let $\{\mathcal{U}(B_1^\sigma, \ldots, B_p^\sigma, U^\sigma); \frac{r\omega}{24R}\}_{\lambda\in\Sigma_k}$ be a set of $\frac{r\omega}{24R}$-orbit-balls that cover $\Gamma_R(y_1, \ldots, y_p, u :$ $v_1, \ldots, v_s,$
$w_1, \ldots, w_t; m, k, \varepsilon)$ with the cardinality of $\Sigma_k$ satisfying

$$|\Sigma_k| = \nu(\Gamma_R(y_1, \ldots, y_p, u : v_1, \ldots, v_s, w_1, \ldots, w_t; m, k, \varepsilon); \frac{r\omega}{24R}).$$

When $m$ is sufficiently large and $\varepsilon$ is sufficiently small, by Theorem 2.1 in [2], we can assume that all $U^\lambda, U^\sigma$ to be Haar unitary matrices in $\mathcal{M}_k(\mathbb{C})$.

For any

$$(A_1, \ldots, A_n, B_1, \ldots, B_p, U) \in \Gamma_R(x_1, \ldots, x_n, y_1, \ldots, y_p, u : v_1, \ldots, v_s, w_1, \ldots, w_t; m, k, \varepsilon),$$

there exist some $\lambda \in \Lambda_k$, $\sigma \in \Sigma_k$ and $W_1, W_2 \in \mathcal{U}_k$ such that

$$\|(A_1, \ldots, A_n, U) - W_1(A_1^\lambda, \ldots, A_n^\lambda, U^\lambda)W_1^*\|_2 \le \frac{r\omega}{24R},$$

$$\|(B_1, \ldots, B_p, U) - W_2(B_1^\sigma, \ldots, B_p^\sigma, U^\sigma)W_2^*\|_2 \le \frac{r\omega}{24R}.$$

Therefore

$$\|W_1U^\lambda W_1^* - W_2U^\sigma W_2^*\|_2 = \|W_2^*W_1U^\lambda - U^\sigma W_2^*W_1\|_2 < \frac{r\omega}{12R}.$$

From Lemma 3.9, there exists a set $\{Ball(U_{\lambda,\sigma,\gamma}, \frac{\omega}{3R})\}_{\gamma\in\Delta_k}$ in $\mathcal{U}_k$ which cover $\Omega(U^\lambda, U^\sigma; \frac{r\omega}{12R})$ with cardinality $|\Delta_k| \le (\frac{18R}{\omega})^{4rk^2}$. This implies

$$\begin{aligned} & \|(A_1, \ldots, A_n, B_1, \ldots, B_p, U) \\ - & (W_2U_{\lambda,\sigma,\gamma}A_1^\lambda U_{\lambda,\sigma,\gamma}^*W_2^*, \ldots, W_2U_{\lambda,\sigma,\gamma}A_n^\lambda U_{\lambda,\sigma,\gamma}^*W_2^*, W_2B_1^\sigma W_2^*, \ldots, W_2B_p^\sigma W_2^*, W_2U^\sigma W_2^*)\|_2 \\ \le & n\omega. \end{aligned}$$

Then we get

$$\begin{aligned}
& \mathfrak{K}(x_1,\ldots,x_n,y_1,\ldots,y_p,u:v_1,\ldots,v_s,w_1,\ldots,w_t;2n\omega,R) \\
\leq\ & \inf_{m\in\mathbb{N},\varepsilon>0}\limsup_{k\to\infty}\frac{\log(|\Lambda_k||\Sigma_k||\Delta_k|)}{-k^2\log(2n\omega)} \\
\leq\ & 4r\frac{\log(18R)-\log\omega}{-\log(2n\omega)}.
\end{aligned}$$

Because $r$ is an arbitrarily small positive number, we have

$$\mathfrak{K}_3(x_1,\ldots,x_n,y_1,\ldots,y_p,u:v_1,\ldots,v_s,w_1,\ldots,w_t)=0.$$

Note that $W^*(x_1,\ldots,x_n,y_1,\ldots,y_p,u)=W^*(x_1,\ldots,x_n,y_1,\ldots,y_p)$, by Lemma 3.7, we have

$$\mathfrak{K}_3(x_1,\ldots,x_n,y_1,\ldots,y_p:v_1,\ldots,v_s,w_1,\ldots,w_t)=0.$$

□

Now we are ready to prove the following theorem.

**Theorem 3.12** *Suppose $\mathcal{M}$ is a von Neumann algebra with a tracial state $\tau$, $\mathcal{N}_1$ and $\mathcal{N}_2$ are von Neumann subalgebras of $\mathcal{M}$. If $\mathcal{N}_1\cap\mathcal{N}_2$ is diffuse, then*

$$\mathfrak{K}_3((\mathcal{N}_1\cup\mathcal{N}_2)'')\leq\mathfrak{K}_3(\mathcal{N}_1)+\mathfrak{K}_3(\mathcal{N}_2).$$

Proof. If one of $\mathfrak{K}_3(\mathcal{N}_1)$ and $\mathfrak{K}_3(\mathcal{N}_2)$ is infinity, then we are done.

Now suppose $\mathfrak{K}_3(\mathcal{N}_1)=\mathfrak{K}_3(\mathcal{N}_2)=0$ and $u$ is a Haar unitary in $\mathcal{N}_1\cap\mathcal{N}_2$. Let $\mathcal{G}=\mathcal{N}_1\cup\mathcal{N}_2$ and $A_0=\{u\}$. Then $\mathcal{G}$ is a generating set of $\mathcal{M}$. Suppose $A_0\subseteq A\subseteq\mathcal{G}$ and $A$ is finite, write $A=\{x_1,\ldots,x_n,u,y_1,\ldots,y_p\}$ with $x_1,\ldots,x_n\in\mathcal{N}_1$ and $y_1,\ldots,y_p\in\mathcal{N}_2$. Since $\mathfrak{K}_3(\mathcal{N}_1)=\mathfrak{K}_3(\mathcal{N}_2)=0$, there exist $v_1,\ldots,v_s\in\mathcal{N}_1$, $w_1,\ldots,w_t\in\mathcal{N}_2$ such that $\mathfrak{K}_2(x_1,\ldots,x_n,u:v_1,\ldots,v_s)=0$ and $\mathfrak{K}_2(y_1,\ldots,y_p,u:w_1,\ldots,w_t)=0$.

Because $u\in W^*(x_1,\ldots,x_n,u)\cap W^*(y_1,\ldots,y_p,u)$, then from Lemma 3.11, we know that

$$\begin{aligned}
& \mathfrak{K}_2(A:v_1,\ldots,v_s,w_1,\ldots,w_t) \\
=\ & \mathfrak{K}_2(x_1,\ldots,x_n,u,y_1,\ldots,y_p:v_1,\ldots,v_s,w_1,\ldots,w_t) \\
=\ & 0.
\end{aligned}$$

Therefore, by Theorem 3.1, $\mathfrak{K}_3((\mathcal{N}_1\cup\mathcal{N}_2)'')=0$. □

**Lemma 3.13** *([1], Lemma 17) Suppose $\mathcal{M}$ is a von Neumann algebra with a tracial state $\tau$, $x$ is a normal element in $\mathcal{M}$ such that $x$ has no eigenvalues. Then there is a selfadjoint element $y$ with the uniform distribution on $[0,1]$ such that $W^*(x)=W^*(y)$.*

**Lemma 3.14** *([1], Lemma 18) Suppose $n,n_1,p\in\mathbb{N}$, $1\leq n_1\leq n$ and $p\geq 2$. Suppose $A$ is a diagonal matrix whose diagonal entries are $\frac{1}{n_1},\frac{2}{n_1},\cdots,\frac{n_1}{n_1},-1,\cdots,-1$, and $B$ is any selfadjoint $n\times n$ matrix with $0\leq B\leq 1$. Suppose $0\leq\varepsilon\leq\frac{1}{4}$, and $\frac{1}{n_1}<4\varepsilon^{p-1}$. Let*

$$\Sigma(A,B,\varepsilon^p)=\{W\in\mathcal{M}_n:\|W\|\leq 1,\|AW-WB_2\|_2<\varepsilon^p\}.$$

*Then $\nu_2(\Sigma(A,B,\varepsilon^p),\varepsilon)\leq(\frac{6}{\varepsilon})^{16n^2\varepsilon^{p-1}}$.*

**Theorem 3.15** *Let $\mathcal{M}$ be a von Neumann algebra with a tracial state $\tau$. Suppose $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ and $y$ is an element in $\mathcal{M}$. If $a$, $b$ are normal operators in $\mathcal{N}$ such that $a$ and $b$ have no common eigenvalues and $ay = yb \neq 0$, then $\mathfrak{K}_3\left(W^*(\mathcal{N} \cup \{y\})\right) \leq \mathfrak{K}_3(\mathcal{N})$.*

Proof. There is no loss in assuming that $y, a, b$ have norm at most 1.

If $\mathfrak{K}_3(\mathcal{N})$ is infinity, then we are done. So we can assume $\mathfrak{K}_3(\mathcal{N}) = 0$.

Since $ay = yb$, by the Putnam-Fuglede theorem, we have $a^*y = yb^*$. Then for any polynomial $p$, $p(a, a^*)a = ap(b, b^*)$. Therefore for every bounded Borel function $\varphi : \mathbb{C} \to \mathbb{C}$, $\varphi(a)y = y\varphi(b)$. If $\lambda \in \sigma_p(a)$, then the spectral projection $\chi_{\{\lambda\}}(b)$ is 0. If $a$ is diagonal, then $y = Iy = \sum_{\lambda\in\sigma_p(a)} \chi_{\{\lambda\}}(a)y = y\sum_{\lambda\in\sigma_p(a)} \chi_{\{\lambda\}}(b) = 0$. So $a$ can not be diagonal, therefore we can write $a = d \oplus c$, where $d$ is diagonal and $c$ has no eigenvalues. From Lemma 3.13, there is a self-adjoint element $c_0$ with the uniform distribution such that $W^*(c) = W^*(c_0)$, thus there is some Borel function $\psi$ such that $c_0 = \psi(c)$ and $\psi(d) = -1$, $0 \leq \psi(b) \leq 1$. Since $\psi(a)y = (-1 \oplus c_0)y = y\psi(b)$, we can replace $-1 \oplus c_0$ with $a$, $\psi(b)$ with $b$. Hence we can assume $b$ is selfadjoint and $a = -1 \oplus c_0$ with $\tau(a^n) = (-1)^n(1 - \alpha) + \alpha \int_0^1 t^n dt$, where $\alpha = 1 - \tau(1 \oplus 0)$.

For each $n \in \mathbb{N}$, define the diagonal matrix $A_n$ with eigenvalues

$$\frac{1}{[n\alpha]}, \frac{2}{[n\alpha]}, \ldots, \frac{[n\alpha]}{[n\alpha]}, -1, -1, \ldots, -1,$$

where $[n\alpha]$ denotes the greatest integer function of $n\alpha$. It is not hard to show that $\tau_n(A_n^m) = (-1)^m \frac{n-[n\alpha]}{n} + \frac{[n\alpha]}{m}\left(\frac{1}{[n\alpha]}\sum_{s=1}^{[n\alpha]}(\frac{s}{[n\alpha]})^m\right)$. When $n \to \infty$, $\frac{1}{[n\alpha]}\sum_{s=1}^{[n\alpha]}(\frac{s}{[n\alpha]})^m$ is a Riemann sum converging to $\int_0^1 t^m dt$. Hence $\tau_n(f(A_n)) \to \tau(f(a))$ as $n \to \infty$. Choose matrix $B_n \in \mathcal{M}_n(\mathbb{C})$ such that $0 \leq B_n \leq 1$ and $B_n$ converges in distribution to $b$ as $n \to \infty$.

Let $x_1, \ldots, x_n$ be elements in $\mathcal{N}$. Then there exist $y_1, \ldots, y_p$ in $\mathcal{N}$ such that

$$\mathfrak{K}_2(x_1, \ldots, x_n : y_1, \ldots, y_p) = 0.$$

For any $0 < \omega < 1$, $0 < r < 1$, $R > 1$, $m \in \mathbb{N}$, $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists a set $\{\mathcal{U}(T_1^\lambda, \ldots, T_n^\lambda, A^\lambda, B^\lambda;$
$\frac{r\omega}{64})\}_{\lambda\in\Lambda_k}$ of $\frac{r\omega}{64}$-orbit-balls in $\mathcal{M}_k(\mathbb{C})^{k+2}$ that cover $\Gamma_R(x_1, \ldots, x_n, a, b : y_1, \ldots, y_p; m, k, \varepsilon)$ with the cardinality of $\Lambda_k$ satisfying $|\Lambda_k| = \nu(\Gamma_R(x_1, \ldots, x_n, a, b : y_1, \ldots, y_p; m, k, \varepsilon), \frac{r\omega}{64})$. When $m$ is sufficiently large and $\varepsilon$ is sufficiently small, we can assume that $A^\lambda$ to be $A_k$ and $B^\lambda$ to be $(U^\lambda)^* B_k U^\lambda$ for some unitary matrix $U$.

For $m$ is sufficiently large and $\varepsilon$ $(\leq \frac{r\omega}{64})$ is sufficiently small, when $(T_1, \ldots, T_n, A, B, C) \in \Gamma_R(x_1, \ldots, x_n,$
$a, b, y : y_1, \ldots, y_p; m, k, \varepsilon)$, it follows from Lemma 4 in [1] that we can assume that $\|C\| \leq 1$. In addition, it is clear that $\|AC - CB\|_2 \leq \varepsilon$ and $(T_1, \ldots, T_n, A, B) \in \Gamma_R(x_1, \ldots, x_n, a, b : y_1, \ldots, y_p; m, k, \varepsilon)$. So there exist some $\lambda \in \Lambda_k$ and $V \in \mathcal{U}_k$ such that

$$\|(T_1, \ldots, T_n, A, B) - (VT_1^\lambda V^*, \ldots, VT_n^\lambda V^*, VA_kV^*, V(U^\lambda)^* B_k U^\lambda V^*)\|_2 \leq \frac{r\omega}{64}.$$

Hence

$$\|A_kV^*CV - V^*CVU^*B_kU\|_2 = \|VA_kV^*C - CV(U^\lambda)^*B_kU^\lambda V^*\|_2 \leq \frac{r\omega}{16}.$$

Then, by Lemma 3.14, there exists a set $\{Ball(C_\sigma; \frac{\omega}{4})\}_{\sigma\in\Sigma_k}$ of $\frac{\omega}{4}$-balls that cover $\{W \in \mathcal{M}_k : \|W\| \leq 1, \ \|WA_k - (U^\lambda)^* B_k U^\lambda W\|_2 < \frac{r\omega}{16}\}$ with $|\Sigma_k| \leq (\frac{24}{\omega})^{32rk^2}$, i.e., there exists some $C_\sigma$ such that

$$\|V^*CV - C_\sigma\|_2 = \|C - VC_\sigma V^*\|_2 \leq \frac{\omega}{4}.$$

Thus

$$\|(T_1, \ldots, T_n, C) - (VT_1^\lambda V^*, \ldots, VT_n^\lambda V^*, VC_\sigma V^*)\|_2 \leq \frac{\omega}{2}.$$

Therefore

$$\nu(\Gamma_R(x_1, \ldots, x_n, y : a, b, y_1, \ldots, y_p); \omega) \leq |\Lambda_k| \cdot |\Sigma_k|.$$

Hence, we get

$$\begin{aligned}
0 &\leq \mathfrak{K}(x_1, \ldots, x_n, y : a, b, y_1, \ldots, y_p; \omega, R) \\
&\leq \inf_{m\in\mathbb{N},\varepsilon>0} \limsup_{k\to\infty} \frac{\log(|\Lambda_k||\Sigma_k|)}{-k^2 \log \omega} \\
&\leq \inf_{m\in\mathbb{N},\varepsilon>0} \limsup_{k\to\infty} \left( \frac{\log(|\Lambda_k|)}{-k^2 \log \omega} + \frac{32rk^2(\log 24 - \log \omega)}{-k^2 \log \omega} \right) \\
&= \inf_{m\in\mathbb{N},\varepsilon>0} \limsup_{k\to\infty} \frac{\log(|\Lambda_k|)}{-k^2 \log \omega} + 32r \frac{\log 24 - \log \omega}{-\log \omega} \\
&= 32r \frac{\log 24 - \log \omega}{-\log \omega}.
\end{aligned}$$

Since $r$ is an arbitrarily small positive number, we have $\mathfrak{K}(x_1, \ldots, x_n, y : a, b, y_1, \ldots, y_p; \omega, R) = 0$, whence, $\mathfrak{K}_2(x_1, \ldots, x_n, y : a, b, y_1, \ldots, y_p) = 0$. Therefore, by theorem 3.1, $\mathfrak{K}_3(\mathcal{M}) = 0$. □

**Corollary 3.16** *Let $\mathcal{M}$ be a von Neumann algebra with a faithful trace $\tau$. Suppose $a, x_1, \ldots, x_n$ are elements in $\mathcal{M}$ such that $a$ is a normal element without eigenvalues, and $ax_i = x_i a$ for all $1 \leq i \leq n$. Then*

$$\mathfrak{K}_3(x_1, \ldots, x_n : u) = 0$$

Using the similar idea in the proof of Theorem 3.15, we can prove the following theorem.

**Theorem 3.17** *Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful trace $\tau$ and $\mathcal{M} = \{\mathcal{N}, u\}''$, where $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$, $u$ is a unitary element of $\mathcal{M}$. Let $\{v_1, v_2, \ldots\}$ be a family of Haar unitary elements and $\{w_1, w_2, \ldots\}$ be a family of unitary elements in $\mathcal{N}$ such that $\|v_n u - uw_n\|_2 \to 0$. Then $\mathfrak{K}_3(\mathcal{M}) \leq \mathfrak{K}_3(\mathcal{N})$.*

*In particular, if $v$ and $w$ are Haar unitary elements in $\mathcal{N}$ such that $vu = uw$, then $\mathfrak{K}_3(\mathcal{M}) \leq \mathfrak{K}_3(\mathcal{N})$.*

# 4 Applications

Suppose $\Lambda$ is an infinite set. An *ultrafilter* $\alpha$ on $\Lambda$ is a collection of subsets of $\mathbb{N}$ such that the empty set $\emptyset \notin \alpha$, $\alpha$ is closed under finite intersections, and, for each subset $A$ of $\Lambda$, either $A \in \alpha$ or $\mathbb{N} \setminus A \in \alpha$. One example of an ultrafilter is obtained by choosing an $\iota$ in $\Lambda$ and letting $\alpha$ be the collection of all subsets of $\Lambda$ that contain $\iota$. Such an ultrafilter is called *principal*; ultrafilters

not of this form are called *free*. We will call an ultrafilter $\alpha$ *nontrivial* if it is free and there exists a decreasing sequence in $\alpha$ whose intersection is empty. Free ultrafilters on an countable set are always nontrivial.

Suppose $\mathfrak{X}$ is another set, $f : \Lambda \to \mathfrak{X}$ is a mapping and $E \subseteq \mathfrak{X}$. we say that $f(\iota)$ is *eventually in* $E$ along $\alpha$ if $f^{-1}(E) = \{\iota \in \Lambda : f(\iota) \in E\} \in \alpha$. If $\mathfrak{X}$ is a topological space, we say that $f(\iota)$ *converges to* $x$ (in $\mathfrak{X}$) along $\alpha$, denoted by $\lim_{\iota\to\alpha} f(\iota) = x$, if $f(\iota)$ is eventually in each neighborhood of $x$. It is well known that if $\mathfrak{X}$ is a compact Hausdorff space, the $\lim_{\iota\to\alpha} f(\iota)$ always exists in $\mathfrak{X}$ for every $f : \Lambda \to \mathfrak{X}$ and every ultrafilter $\alpha$ on $\Lambda$.

Let $\alpha$ be a nontrivial ultrafilter on $\Lambda$. Suppose $\mathcal{M}_\iota$ is a finite von Neumann algebra with a faithful trace $\tau_\iota$. Let $\prod_\iota \mathcal{M}_\iota$ be the $l^\infty$-product of the $\mathcal{M}_\iota$'s, $\mathcal{J} = \{\{x_\iota\} : \lim_{\iota\to\alpha} \tau_\iota(x_\iota^* x_\iota) = 0\}$. Then define the *ultraproduct* $\prod^\alpha \mathcal{M}_\iota$ of $\mathcal{M}_\iota$ to be $\prod_{i\in\mathbb{I}} \mathcal{M}_i/\mathcal{J}$.

When $\mathcal{M}_\iota = \mathcal{M}$ for all $\iota$, then $\prod^\alpha \mathcal{M}_\iota$ is called the *ultrapower* of $\mathcal{M}$, denoted by $\mathcal{M}^\alpha$.

Let $\mathcal{M}$ be a $\mathrm{II}_1$ factor with the faithful trace $\tau$. For every $\varepsilon > 0$, and any elements $x_1, x_2, \ldots, x_n$ in $\mathcal{M}$, if there exists a unitary $u \in \mathcal{M}$ with $\tau(u) = 0$ such that $\|ux_i - x_iu\|_2 \le \varepsilon$ for every $i$, then we say that $\mathcal{M}$ has *property* $\Gamma$.

It is well-known that if $\mathcal{M}$ is a $\mathrm{II}_1$ factor with property $\Gamma$, then there exists a central sequence $\{v_n\}$ of Haar unitary elements in $\mathcal{M}$, i.e., $\|v_n x - x v_n\|_2 \to 0$ for every $x \in \mathcal{M}$.

If a von Neumann algebra acts on a very large Hilbert space, it may not contain any nontrivial central sequences, but it may contain a *central net*, i.e., a net $\{x_\lambda\}$ in $\mathcal{M}$ such that $\|x_\lambda a - a x_\lambda\|_2 \to 0$ for every $a \in \mathcal{M}$. Equivalently, $\mathcal{M}$ has a central net if and only if, for every $\varepsilon > 0$ and for every finite subset $F \subseteq \mathcal{M}$, there is a Haar unitary $u$ such that $\|ua - au\|_2 < \varepsilon$ for every $a \in F$.

**Theorem 4.1** *Suppose $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ is a family of von Neumann algebras such that each $\mathcal{M}_\lambda$ has a central net of Haar untiaries. Let $\alpha$ be a nontrivial ultrafilter on $\Lambda$. Then*

$$\mathfrak{K}_3\Big(\prod^{\alpha} \mathcal{M}_\lambda\Big) = 0.$$

Proof. Suppose $x_1, \ldots, x_n$ are any elements in $\prod^\alpha \mathcal{M}_\lambda$, and $x_i = \{x_\lambda^i\}_\alpha$. Since $\alpha$ is nontrivial, there exists a decreasing sequence $\{A_k\}_{k=1}^\infty$ in $\alpha$ whose intersection is empty, and $A_1 = \Lambda$. If $\lambda \in A_k/A_{k-1}$, since $\mathcal{M}_\lambda$ has a central net of Haar unitaries, then there exists a Haar unitary $u_\lambda \in \mathcal{M}_\lambda$ such that $\|x_\lambda^i u_\lambda - u_\lambda x_\lambda^i\|_2 < \frac{1}{k}$ for $1 \le i \le n$. Then $u = \{u_\lambda\}_\alpha$ defines a Haar unitary that commutes with $x_1, \ldots, x_n$. By Corollary 3.16, $\mathfrak{K}_2(x_1, \ldots, x_n : u) = 0$. Therefore, by Theorem 3.1, $\mathfrak{K}_3\left(\prod^\alpha \mathcal{M}_\lambda\right) = 0$. $\square$

**Remark 4.2** *Suppose $\mathcal{M}$ is a diffuse finite von Neumann algebra with a faithful trace $\tau$. We can define a numerical invariant $\gamma(\mathcal{M})$ by*

$$\gamma(\mathcal{M}) = \sup_{\substack{F \subset \mathit{ball}(\mathcal{M}) \\ F \text{ finite}}} \ \inf_{\substack{u \text{ is a} \\ \text{Haar unitary}}} \ \max_{a\in F} \|au - ua\|.$$

*It is clear that $\gamma(\mathcal{M}) = 0$ if and only if $\mathcal{M}$ has a central net of Haar unitaries and that $\gamma(\mathcal{M}) \le 2$. F. Murray and J. von Neumann [9] found a lower bound for $\gamma(\mathcal{L}_{\mathbb{F}_n})$ for $n \ge 2$. (Of course, they did not use our terminology.) It is not difficult to modify the proof of Theorem 4.1 to prove that if $\gamma(\mathcal{M}_\lambda) \to 0$ along the ultrafilter $\alpha$, then $\mathfrak{K}_3(\prod^\alpha \mathcal{M}_\lambda) = 0$.*

**Lemma 4.3** *[3] Suppose $\mathcal{M}$ is diffuse, countably generated and has a central sequence of Haar unitaries. Then there is a central sequence $\{u_n\}$ of mutually commuting Haar unitaries in $\mathcal{M}$.*

**Theorem 4.4** *If $\mathcal{M}$ is a von Neumann algebra with a central net of Haar unitaries, then $\mathfrak{K}_3(\mathcal{M}) = 0$.*

Proof. Suppose $x_1, \ldots, x_n \in \mathcal{M}$. Then there is a sequence $\{u_n\}$ of Haar unitaries so that if $a \in \{x_1, \ldots, x_n, u_1, u_2, \ldots\}$, then

$$\|au_n - u_n a\|_2 \to 0,$$

i.e., inductively choose $u_n$ so that $\|au_n - u_n a\|_2 < 1/n$ for $a \in \{x_1, \ldots, x_n, u_1, u_2, \ldots u_{n-1}\}$. Hence $\{u_n\}$ is a central sequence in the von Neumann algebra $\mathcal{N}$ generated by $\{x_1, \ldots, x_n, u_1, u_2, \ldots\}$. It follows from Lemma 4.3 that there is a central sequence $\{v_n\}$ of commuting Haar unitaries in $\mathcal{N}$. Then we get $\mathfrak{K}_3\left(\{v_1, v_2, \ldots\}''\right) = 0$. We can choose unitaries $\{w_1, \ldots, w_m\}$ that generate $W^*(x_1, \ldots, x_n)$, and, using Theorem 3.17, we inductively get $\mathfrak{K}_3\left(\{w_1, \ldots, w_j, v_1, v_2, \ldots\}''\right) = 0$ for $1 \leq j \leq m$. Hence, there exist $y_1, \ldots, y_p \in \mathcal{N}$ such that

$$\mathfrak{K}_2(x_1, \ldots, x_n : y_1, \ldots, y_p) = 0.$$

Therefore $\mathfrak{K}_3(\mathcal{M}) = 0$. $\square$

**Theorem 4.5** *Suppose $\mathbb{F}$ is a free group with the standard generating set $G$ satisfying $|G| \geq 2$ and let $\mathcal{L}_{\mathbb{F}}$ be the group von Neumann algebra generated by $\mathbb{F}$. Then $\mathfrak{K}_3(\mathcal{L}_{\mathbb{F}}) = \infty$.*

Proof. For any $g \in \mathbb{F}$, we can view $g$ as a unitary in $\mathcal{L}_{\mathbb{F}}$. Note that $G$ generates $\mathcal{L}_{\mathbb{F}}$. Let $g_1, \ldots, g_n \in G$ $(n \geq 2)$, and $y_1, y_2, \ldots, y_N \in G$. We will prove that $n \leq \delta(g_1, \ldots, g_n : y_1, y_2, \ldots, y_N)$. Since

$$\delta(g_1, \ldots, g_n : y_1, y_2, \ldots, y_N) \leq 1 + \mathfrak{K}_2(g_1, \ldots, g_n : y_1, y_2, \ldots, y_N),$$

we will conclude that $n - 1 \leq \mathfrak{K}_2(g_1, \ldots, g_n : y_1, y_2, \ldots, y_N)$. From this it follows that $\mathfrak{K}_3(\mathcal{L}_{\mathbb{F}}) = \mathfrak{K}_3(G) = \infty$.

It follows from Theorem 13 in [1] that when we compute $\delta$ we can replace the $\Gamma_R$-sets with the set of unitary elements in the $\Gamma_R$-sets. Let

$$\Omega_{m,k,\varepsilon} = \mathcal{U}_k^{n+N} \cap \Gamma_R(g_1, \ldots, g_n, y_1, y_2, \ldots, y_N; m, k, \varepsilon),$$

$$\Delta_{m,k,\varepsilon} = \mathcal{U}_k^{n} \cap \Gamma_R(g_1, \ldots, g_n : y_1, y_2, \ldots, y_N; m, k, \varepsilon).$$

If $(U_1, \ldots, U_n, W_1, \ldots, W_N) \in \Omega_{m,k,\varepsilon}$, then $(U_1, \ldots, U_n) \in \Delta_{m,k,\varepsilon}$. So

$$\Omega_{m,k,\varepsilon} \subseteq \Delta_{m,k,\varepsilon} \times \mathcal{U}_k^N.$$

Let $\mu_k$ denote Haar measure on $\mathcal{U}_k$, $\mu_k^n$ denote the corresponding product measure on $\mathcal{U}_k^n$. Then

$$\mu_k^{n+N}(\Omega_{m,k,\varepsilon}) \leq \mu_k^n(\Delta_{m,k,\varepsilon}) \cdot \mu_k^N\left(\mathcal{U}_k^N\right) \leq \mu_k^n(\Delta_{m,k,\varepsilon}) \leq 1.$$

We know from Theorem 3.9 in [14] that $\mu_k^{n+N}(\Omega_{m,k,\varepsilon}) \to 1$ and thus $\mu_k^n(\Delta_{m,k,\varepsilon}) \to 1$ as $k \to \infty$. It follows that

$$\mu_k^n(\Delta_{m,k,\varepsilon}) \leq \nu_2(\Delta_{m,k,\varepsilon}, \omega)\mu_k^n(ball(1,\omega)) \leq \nu_2(\Delta_{m,k,\varepsilon}, \omega)(\omega)^{nk^2},$$

so $\nu_2(\Delta_{m,k,\varepsilon},\omega) \geq \mu_k^n(\Delta_{m,k,\varepsilon})(\frac{1}{\omega})^{nk^2}$.

By Lemma 1 in [4],

$$\mathfrak{K}_2(g_1,\ldots,g_n : y_1,\ldots,y_N) \geq \delta(g_1,\ldots,g_n : y_1,\ldots,y_N) - 1.$$

Note that

$$\begin{aligned}
\delta(g_1,\ldots,g_n : y_1,\ldots,y_N) &= -1 + \limsup_{\omega\to 0^+} \inf_{m,\varepsilon} \limsup_{k\to\infty} \frac{\log\left(\nu_2(\Delta_{m,k,\varepsilon},\omega)\right)}{-k^2\log\omega} \\
&\geq -1 + \limsup_{\omega\to 0^+} \inf_{m,\varepsilon} \limsup_{k\to\infty} \frac{\log\left(\mu_k^n(\Delta_{m,k,\varepsilon})\right) - nk^2\log\omega}{-k^2\log\omega}.
\end{aligned}$$

Since $\mu_k^n(\Delta_{m,k,\varepsilon}) \to 1$, we have

$$\mathfrak{K}_2(g_1,\ldots,g_n : y_1,\ldots,y_N) \geq n-1.$$

Thus $\mathfrak{K}_3(\mathcal{L}_G) = \infty$. □

**Theorem 4.6** *Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful trace $\tau$, $\mathcal{N}_1$ and $\mathcal{N}_2$ are mutually commuting diffuse subalgebras of $\mathcal{M}$. Then $\mathfrak{K}_3(W^*(\mathcal{N}_1\cup\mathcal{N}_2)) = 0$.*

Proof. Since $\mathcal{N}_1$ and $\mathcal{N}_2$ are diffuse, we can assume that $\mathcal{N}_1 = \{u_\lambda : \lambda\in\Lambda\}''$ and $\mathcal{N}_2 = \{v_\sigma : \sigma\in\Sigma\}''$ where $u_\lambda, v_\sigma$ are all Haar unitaries. For any finite subset $E$ of $\Lambda$ and finite subset $F$ of $\Sigma$, let $\mathcal{M}_{E,F} = \{u_\lambda, v_\sigma : \lambda\in E, \sigma\in F\}''$. Then $\mathfrak{K}_3(\mathcal{M}_{E,F}) = 0$ by Theorem 3.17. Let

$$\mathcal{S} = \bigcup\left\{\mathcal{M}_{E,F} : E \text{ is a finite subset of } \Lambda, F \text{ is a finite subset of } \Sigma\right\}.$$

It is clear that $\mathcal{S}$ generates $W^*(\mathcal{N}_1\cup\mathcal{N}_2)$. By Corollary 3.4, $\mathfrak{K}_3(\mathcal{S}) = 0$. Thus $\mathfrak{K}_3(W^*(\mathcal{N}_1\cup\mathcal{N}_2)) = 0$ by Corollary 3.3. □

**Corollary 4.7** *If $\mathcal{N}_1$ and $\mathcal{N}_2$ are diffuse von Neumann algebras, then $\mathfrak{K}_3(\mathcal{N}_1\otimes\mathcal{N}_2) = 0$.*

The following corollary was proved by S. Popa [11] and L. Ge [5].

**Corollary 4.8** *If $\mathcal{M}$ is a $II_1$ factor and $\mathfrak{K}_3(\mathcal{M}) = \infty$, then $\mathcal{M}$ is prime. In particular, $\mathcal{L}_{\mathbb{F}}$ is prime for every free group $\mathbb{F}$ with the standard generating set $G$ satisfying $|G|\geq 2$..*

Let $\mathcal{M}$ be a von Neumann algebra with a faithful trace, $\mathcal{U}(\mathcal{M})$ be the set of all unitary elements in $\mathcal{M}$. For any subset $\mathcal{S}\subseteq\mathcal{M}$, define

$$\mathcal{N}(\mathcal{S}) = \{u\in\mathcal{U}(\mathcal{M}) : u\mathcal{S}u^*\subseteq\mathcal{S}\}'',$$

and

$$\begin{aligned}
\mathcal{I}(\mathcal{S}) &= W^*(\{y\in\mathcal{M} : \exists \text{ two normal elements } a,b \\
&\qquad \text{without common eigenvalues such that } ay = yb \neq 0\})
\end{aligned}$$

Suppose $\mathcal{A}$ is a diffuse von Neumann subalgebra of $\mathcal{M}$, and $\alpha$ is an ordinal. Then define

$$\mathcal{N}_\alpha(\mathcal{A}) = \begin{cases} \mathcal{A} & \alpha = 0 \\ \left(\bigcup_{\beta<\alpha} \mathcal{N}_\beta(\mathcal{A})\right)'' & \alpha \text{ is a limit ordinal} \\ \mathcal{N}\left(\mathcal{N}_\beta(\mathcal{A})\right) & \text{if } \alpha = \beta + 1, \end{cases}$$

and

$$\mathcal{I}_\alpha(\mathcal{A}) = \begin{cases} \mathcal{A} & \alpha = 0 \\ \left(\bigcup_{\beta<\alpha} \mathcal{I}_\beta(\mathcal{A})\right)'' & \alpha \text{ is a limit ordinal} \\ \mathcal{I}\left(\mathcal{I}_\beta(\mathcal{A})\right) & \text{if } \alpha = \beta + 1. \end{cases}$$

The following theorem is a easy consequence of Theorem 3.17, Theorem 3.15 and Corollary 3.4.

**Theorem 4.9** *Let $\mathcal{M}$ be a von Neumann algebra with faithful trace $\tau$ and $\mathcal{A}$ be a diffuse subalgebra of $\mathcal{M}$ with $\mathfrak{K}_3(\mathcal{A}) = 0$. Then for any ordinal $\alpha$,*

$$\mathfrak{K}_3\left(\mathcal{N}_\alpha(\mathcal{A})\right) = \mathfrak{K}_3\left(\mathcal{I}_\alpha(\mathcal{A})\right) = 0.$$

## 5 Applications to group theory

Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful trace $\tau$ and $u$ is a Haar unitary in $\mathcal{M}$. Define

$$\mathcal{N}_u = (\cup\{\mathcal{N} : \mathcal{N} \subseteq \mathcal{M}, u \in \mathcal{N}, \mathfrak{K}_3(\mathcal{N}) = 0\})''$$

to be the von Neumann subalgebra generated by the union of those subalgebras $\mathcal{N}$ containing $u$ such that $\mathfrak{K}_3(\mathcal{N}) = 0$. It follows from Theorem 3.12 that $\mathfrak{K}_3(\mathcal{N}_u) = 0$. Therefore $\mathcal{N}_u$ is the unique largest subalgebra containing $u$ with $\mathfrak{K}_3(\mathcal{N}_u) = 0$.

Suppose $\mathcal{B}$ is a diffuse von Neumann subalgebra of $\mathcal{M}$ with $\mathfrak{K}_3(\mathcal{B}) = 0$. Since $\mathcal{B}$ is diffuse, there exists a Haar unitary $u \in \mathcal{B}$. It is clear that $\mathcal{N}_u$ is the largest subalgebra of $\mathcal{M}$ that contains $\mathcal{B}$ whose $\mathfrak{K}_3$ is 0.

If $u, v$ are two Haar unitaries in $\mathcal{M}$ such that $\mathcal{N}_u \cap \mathcal{N}_v$ is diffuse, then, by Theorem 3.12, $\mathfrak{K}_3\left((\mathcal{N}_u \cup \mathcal{N}_v)''\right) = 0$. Therefore $\mathcal{N}_u = \mathcal{N}_v$ by the maximality of $\mathcal{N}_u$ and $\mathcal{N}_v$. Therefore, if $\mathcal{N}_u \neq \mathcal{N}_v$, then $\mathcal{N}_u \cap \mathcal{N}_v$ is not diffuse.

By Theorem 4.9, it is clear that, for any Haar unitary $u \in \mathcal{M}$, the normalizer of $\mathcal{N}_u$ is $\mathcal{N}_u$.

The above ideas have an interesting interpretation in discrete groups. Let $G$ be a group and $\mathcal{L}_G$ be the corresponding group von Neumann algebra. For any $g \in G$, we can view $g$ as a unitary in $\mathcal{L}_G$. If $g$ is an element in $G$ with infinite order, then $g$ is a Haar unitary in $\mathcal{L}_G$. If $H$ is a subgroup of $G \subseteq \mathcal{L}_G$, then $H'' \cong \mathcal{L}_H$.

Suppose $G$ is a discrete torsion-free group (i.e., the only element of finite order is identity) and $\mathcal{L}_G$ can be embedded into an ultrapower of the hyperfinite $\mathrm{II}_1$ factor. F. Radulescu (see Proposition 2.5 in [13]) proved that this is equivalent to $G$ being algebraically embeddable in the unitary group of such an ultrapower. For any $g \in G \setminus \{e\}$, let $H_g$ be the subgroup generated by the set $\{H \leq G : g \in H, \mathfrak{K}_3(\mathcal{L}_H) = 0\}$.

By Theorem 3.12 and Theorem 3.17, we get the following theorem.

**Theorem 5.1** *Let $G$ be a torsion-free group with the unit $e$. The following statements hold:*

*(1) $\{H_g \setminus \{e\} : g \in G \setminus \{e\}\}$ is a partition of $G \setminus \{e\}$,*

*(2) for any $g \in G \setminus \{e\}$, if $hH_gh^{-1} \subseteq H_g$, then $h \in H_g$,*

*(3) for any $g \in G \setminus \{e\}$, if $h \in G \setminus H_g$, then $hH_g \cap H_gh = \{h\}$.*

We call $\{H_g : g \in G/\{e\}\}$ the $\mathfrak{K}_3$-*decomposition* of the torsion-free group $G$. Determining the $\mathfrak{K}_3$-decomposition of a particular group involves both algebra and the theory of von Neumann algebras. In [4], there are many examples of groups $G$ whose $\mathfrak{K}_3$-decomposition is $\{G\}$. We now provide a particular example.

**Proposition 5.2** *If $G$ is a free group, then, for any $g \in G \setminus \{e\}$, $H_g$ is a maximal cyclic subgroup of $G$ containing $g$. In particular, if $g$ is one of the free generators, then $H_g$ is the subgroup generated by $g$.*

Proof. Since every subgroup of free group is free, $H_g$ is free. By definition of $H_g$ we see that $\mathfrak{K}_3(\mathcal{L}_{H_g}) = 0$. Therefore, by Corollary 4.8, $H_g$ cannot have more than one generator. That implies $H_g$ is cyclic. By Theorem 3.17, $H_g$ must be a maximal abelian subgroup in $G$. □

Let $\mathbb{F}_2$ be a free group generated by two standard generators $u, v$. We know that, in $\mathbb{F}_2$, $H_u$ is the subgroup generated by $u$. This naturally raises the question.

**Question 1**. In $\mathcal{L}_{\mathbb{F}_2}$, is $\mathcal{N}_u = W^*(u)$? In other words, is $W^*(u)$ a maximal subalgebra of $\mathcal{L}_{\mathbb{F}_2}$ whose $\mathfrak{K}_3$ is 0?

S. Popa [12] proved that $W^*(u)$ is maximal injective. An affirmative answer to the question above would imply Popa's result, since $\mathfrak{K}_3(\mathcal{M}) = 0$ whenever $\mathcal{M}$ is injective. This means that answering the question above is likely to be difficult. However, there are natural subquestions based on Theorem 3.12 and Theorem 3.15, respectively.

**Question 1a** If $\mathcal{M}$ is a subalgebra of $\mathcal{L}_{\mathbb{F}_2}$ with $\mathfrak{K}_3(\mathcal{M}) = 0$, and $\mathcal{M} \cap W^*(u)$ is diffuse, then must we have $\mathcal{M} \subseteq W^*(u)$?

**Question 1b** If $y \in \mathcal{L}_{\mathbb{F}_2}$, and $a, b \in W^*(u)$ without common eigenvalues, such that, $ya = by \neq 0$, then must y be in $W^*(u)$?

We can give a partial solution to Question 1b by showing that if $w$ is a unitary in $\mathcal{L}_{\mathbb{F}_2}$ that conjugates a Haar unitary in $W^*(u)$ into $W^*(u)$, then $w \in W^*(u)$.

Suppose that $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras and $\mathcal{N} \subseteq \mathcal{M}$. By a *conditional expectation* from $\mathcal{M}$ onto $\mathcal{N}$, we mean a positive linear mapping $E : \mathcal{M} \to \mathcal{N}$ such that

(1) $E(I) = I$,

(2) $E(x_1yx_2) = x_1E(y)x_2$, for any $x_1, x_2 \in \mathcal{N}$ and $y \in \mathcal{M}$.

Define $\mathcal{A} \perp \mathcal{B}$ to be $\tau(ab) = \tau(a)\tau(b)$ for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

**Theorem 5.3** *Let $u, v$ be standard generators of $\mathcal{L}_{\mathbb{F}_2}$. If $\mathcal{B}$ is a diffuse von Neumann subalgebra of $W^*(u)$, then*

$$\{w : \ w \text{ is a unitary in } \mathcal{L}_{\mathbb{F}_2}, w^*\mathcal{B}w \subseteq W^*(u)\} \subseteq W^*(u).$$

Proof. Suppose $\mathcal{B}$ is a diffuse von Neumann subalgebra of $W^*(u)$. Define

$$\mathcal{N} = \{w : \ w \text{ is a unitary in } \mathcal{L}_{\mathbb{F}_2}, w^*\mathcal{B}w \subseteq W^*(u)\}.$$

It is sufficient to prove $W^*(u)^\perp \subseteq \mathcal{N}^\perp$. The proof is a modification of Lemma 2.5 in [11].

Let $g$ be en element in $W^*(u)^\perp$. It follows that $gW^*(u)g^* \perp W^*(u)$.

Since $\mathcal{B}$ has no atoms, then, for any given $\varepsilon > 0$, there exists an orthogonal family of projections $e_1, \ldots, e_n$ in $\mathcal{B}$ such that $\tau(e_i) < \varepsilon$ for $1 \leq i \leq n$. Let $\mathcal{A}_\varepsilon$ be the von Neumann subalgebra generated by $e_1, \ldots, e_n$, $\tau$ be the unique trace on $\mathcal{L}_{\mathbb{F}_2}$, and $E_{\mathcal{A}'_\varepsilon \cap \mathcal{L}_{\mathbb{F}_2}}$ be the unique $\tau$-preserving conditional expectation from $\mathcal{L}_{\mathbb{F}_2}$ onto $\mathcal{A}'_\varepsilon \cap \mathcal{L}_{\mathbb{F}_2}$ (i.e., $\tau \circ E_{\mathcal{A}'_\varepsilon \cap \mathcal{L}_{\mathbb{F}_2}} = \tau$).

For any $w \in \mathcal{N}$, $g\mathcal{A}_\varepsilon g^* \perp w\mathcal{A}_\varepsilon w^*$ since $g\mathcal{A}_\varepsilon g^* \perp W^*(u)$ and $w\mathcal{A}_\varepsilon w^* \subseteq w\mathcal{B}w^* \subseteq W^*(u)$. Therefore, for $1 \leq i \leq n$,

$$\tau(w^* g e_i g^* w e_i) = \tau(g e_i g^* w e_i w^*) = \tau(g e_i g^*)\tau(w e_i w^*) = \tau(e_i)^2.$$

Summing up over $i$, we get

$$\begin{aligned}
|\tau(wg)|^2 &\leq \|E_{\mathcal{A}'_\varepsilon \cap \mathcal{L}_{\mathbb{F}_2}}(wg)\|_2^2 \\
&= \|\sum_i e_i w g e_i\|_2^2 = \sum_i \|e_i w g e_i\|_2^2 \\
&= \sum_i \tau(w g e_i g^* w^* e_i) = \sum_i \tau(e_i)^2 \\
&\leq (\max_j \tau(e_j)) \sum_i \tau(e_i) < \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily small, $\tau(wg) = 0$. Therefore for any $x \in \mathcal{N}''$, $\tau(x) = 0$. Thus $g \perp \mathcal{N}''$.
□

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